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An analytical study of linear and non-linear double diffusive convection with Soret effect in couple stress liquids

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Abstract

The double diffusive convection in a two-component couple stress liquid layer with Soret effect is studied using both linear and non-linear stability analyses. The linear theory is based on normal mode technique and the non-linear analysis is based on a minimal representation of double Fourier series. The effect of couple stress parameter, the Soret parameter, the solute Rayleigh number, the Prandtl number and the diffusivity ratio on the stationary, oscillatory and finite amplitude convection are shown graphically. It is found that the effects of couple stress are quite large and the positive Soret number enhances the stability while the negative Soret number enhances the instability. The non-linear theory predicts that, finite amplitude motions are possible only for negative Soret parameter. The transient behaviour of thermal and solute Nusselt numbers has been investigated by solving numerically a fifth order Lorenz model using Runge–Kutta method.

Keywords: Double diffusive convection; Soret effect; Couple stress liquid; Double Fourier series; Lorenz model

1. Introduction

The gradients of two stratifying agencies, such as heat and salt, having different diffusivities are simultaneously present in a fluid layer, a variety of interesting convective phenomena can occur which are not possible in a single component fluid. Convection in a fluid layer with two or more stratifying agencies has been the subject of extensive theoretical and experimental investigations in the last few decades. Excellent reviews of these studies have been reported by Turner [1–3], Huppert and Turner [4], Platten and Legros [5]. The interest in the study of two or multi-component convection has developed as a result of the marked difference between single component and multi-component systems. In contrast to single component systems, convection sets in even when density decreases with height, that is, when the basic state is hydrostatically stable. The double diffusive convection is of importance in various fields such as high quality crystal production, liquid gas storage, oceanography, production of pure medication, solidification of molten alloys, and geothermally heated lakes and magmas.

Convection in a two-component fluid is characterized by well-mixed convecting layers, which are separated by relatively sharp density steps. These steps may be of the ‘finger’ or ‘diffusive’ kind and both types of interface must enable a net release of potential energy preferentially transporting the destabilizing property. Salt fingers will occur when warm salty fluid overlies cooler fresher fluid and diffusive instability will occur when warm salty fluid underlies the fresh cooler fluid. In two-component system, in the absence cross-diffusion, instability can occur only if, at least one of the components is destabilizing. However, in the presence of cross-diffusions produced by the simultaneous interference of two transport properties e.g., Soret and Dufour effects the situations may be quite different [6–8]. Hurle and Jakeman [9] argue that for liquid mixtures the Dufour term is indeed very small and thus the Dufour effect will be negligible in comparison to the Soret effect. Hence one may ignore the Dufour term when dealing with flows in liquids.

In most practical situations, where multicomponent fluid flows are involved, cross-diffusion known as the Soret effect is more often the main driver of various convective phenomena that occur within a thermal stratified media. Oscillatory and subcritical motions and hysteresis effects are some examples of these phenomena. Owing to their presence in nature and in en-

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Nomenclature

A_{mn}	amplitudes of streamline perturbation	
B_{mn}	amplitudes of thermal perturbation	
E_{mn}	amplitudes of concentration perturbation	
C	couple stress parameter, $\mu_1/\mu d^2$	
d	height of the fluid layer	m
D	solute diffusivity	$\mathrm{m^2\,s^{-1}}$
D_1	Soret coefficient	$\mathrm{kg\,m^{-1}\,s^{-1}\,K^{-1}}$
D_S	Soret parameter, $D_1\beta_S/\kappa\beta_T$	
$\mathbf{g}$	gravitational acceleration	$\mathrm{m\,s^{-2}}$
H	rate of heat transport per unit area	$\mathrm{m\,s^{-1}\,K}$
J	rate of mass transport per unit area	$\mathrm{m^{-2}\,kg\,s^{-1}}$
K^2	$\pi^2(\alpha^2+1)$	
Nu	thermal Nusselt number,	
Nu^S	solute Nusselt number	
p	pressure	$\mathrm{kg\,m^{-1}\,s^{-2}}$
Pr	Prandtl number, ν/κ	
$\mathbf{q}$	velocity vector, (u, v, w)	$\mathrm{m\,s^{-1}}$
R	thermal Rayleigh number, $\beta_T g\Delta T d^3/\nu\kappa$	
R_S	solute Rayleigh number, $\beta_S g\Delta S d^3/\nu\kappa$	
S	solute concentration	$\mathrm{kg\,m^{-3}}$
ΔS	salinity difference between the walls	$\mathrm{kg\,m^{-3}}$
T	temperature	K
ΔT	temperature difference between the walls	K
t	time	s
x, y, z	space coordinates	m

Greek symbols

α	wavenumber	
β_S	solute expansion coefficient	$\mathrm{kg^{-1}\,m^3}$
β_T	thermal expansion coefficient	$\mathrm{K^{-1}}$
Φ	dimensionless amplitude of concentration perturbation	
η	$1+CK^2$	
κ	thermal diffusivity	$\mathrm{m^2\,s^{-1}}$
μ	dynamic viscosity	$\mathrm{kg\,m^{-1}\,s^{-1}}$
μ_1	couple stress viscosity	$\mathrm{kg\,m\,s^{-1}}$
ν	kinematic viscosity, μ/ρ_0	$\mathrm{m^2\,s^{-1}}$
Θ	dimensionless amplitude of temperature perturbation	
ρ	density	$\mathrm{kg\,m^{-3}}$
σ	growth rate	$\mathrm{s^{-1}}$
τ	diffusivity ratio, D/κ	
ω	frequency	$\mathrm{s^{-1}}$
ψ	stream function	$\mathrm{m^2\,s^{-1}}$
Ψ	dimensionless stream function	

Subscripts

b	basic state
c	critical
0	reference value

Superscripts

$*$	dimensionless quantity
$'$	perturbed quantity
f	finite amplitude
osc	oscillatory
st	stationary

gineering applications such as in geophysics, oil reservoirs, and groundwater, the researchers have developed a great interest in these types of flows. Soret effect, which describes the diffusion of solute along a temperature gradient, within a horizontal layer heated from below has attracted a considerable attention in pattern formations and hysteresis phenomena. In such convective configurations, a negative Soret coefficient, which causes the heavier component to migrate towards the hot wall, induces a variety of pattern formations. Multiple steady oscillatory states, subcritical flow, standing traveling waves and Hopf bifurcations are some examples of these phenomena.

An experimental and theoretical study of Soret driven thermosolutal convection in a binary fluid mixture has been made by Hurle and Jakeman [9]. They found that when the water–methanol mixture is heated from below, initially the oscillatory flow was observed and later it was bifurcated towards the finite amplitude motion. Oscillatory motions in Benard cell due to Soret effect has been studied by Platten and Chavepeyer [10]. They reported that as predicted by the linear theory oscillatory motions are observed in the two-component system with negative Soret coefficients and the order of magnitude of the period of oscillations is confirmed by experiments. Lawson and Yang [11] have investigated the Benard problem in a binary mixture of dilute gases in which an imposed vertical temperature induces a concentration gradient owing to the thermal diffusion effect. Their numerical results indicate that instability will set in only as stationary convection. This is distinctly different from the case of liquids and concentrated gases, in which the Soret effect gives rise to oscillatory instability.

Thermal convection in a binary fluid driven by the Soret and Dufour effects has been investigated by Knobloch [12]. He has shown that equations are identical to the thermosolutal problem except for a relation between the thermal and solute Rayleigh numbers. Lee et al. [13] have studied experimentally the onset of Rayleigh–Benard convection in liquid mixtures. They reported that the overstable motions are possible when heated from below and only stationary motions exist when heated from above. Platten and Legros [5] have carried out a weak non-linear stability analysis of the binary fluid mixtures with stress-free boundaries using the Lorenz model. They determined the threshold of subcritical oscillatory flows. Barten et al. [14,15] have observed the non-linear traveling wave and stationary onset for the negative values of Soret coefficient in the mixtures of binary fluids. They observed the oscillatory convection in a finite container containing the binary fluid systems such as 3He–4He and water–ethanol with realistic boundary conditions. They used the linear stability analysis to find the criteria for the onset of oscillatory convection. A study of convective instability in a fluid mixture heated from above with negative separation ratio (Soret coefficient) was performed experimentally by Porta and Surko [16]. Although the linear analysis predicts that the instability occurs at zero wavenumber, a large wavenumber pattern is observed. The onset is supercritical and convection amplitude exhibits damped oscillations for sudden change in the forcing parameters. Recently, Bourich et al. [17] have given an exhaustive review of literature on the Soret effect convection either in fluid or porous media.

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. During recent years the theory of polar fluids has received much attention and this is because the traditional Newtonian fluids cannot precisely describe the characteristics of the fluid flow with suspended particles. The study of such fluids have applications in a number of processes that occur in industry such as the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath, exotic lubricants and colloidal and suspension solutions. In the category of non-Newtonian fluids couple stress fluid has distinct features, such as polar effects. The theory of polar fluids and related theories are models for fluids whose microstructure is mechanically significant. The constitutive equations for couple stress fluids were given by Stokes [18]. The theory proposed by Stokes [18] is the simplest one for micro fluids, which allows polar effects such as the presence of couple stress, body couples and non-symmetric tensors. Couple stresses are found to appear in noticeable magnitude in fluids with very large molecules. Rayleigh–Benard convection in fluids with stress non-linearly proportional to velocity gradient is studied by few researchers (see e.g. [19–21]). However the study on double diffusive convection in couple stress fluids is not available to the authors' knowledge. Therefore, in the present paper we investigate the effect of Soret coefficient on the onset of double diffusive convection using the Stokes' [18] couple stress model. Further we perform a non-linear stability analysis of the problem using the minimal representation of Fourier series with the object of computing heat and mass transports.

2. Mathematical formulation

2.1. Basic equations

The continuity and momentum equations governing the motion of an incompressible couple stress fluid in the absence of body couple are given by [18]

$$\nabla \cdot \mathbf{q} = 0 \tag{2.1}$$

$$\rho_0\left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q}\right] = -\nabla p + \mu\nabla^2\mathbf{q} - \mu_1\nabla^4\mathbf{q} + \rho\mathbf{g} \tag{2.2}$$

In case of polar fluids the action of one part of the body on its neighborhood cannot be represented by a force alone but rather by a force and couple. The third term on the right-hand side of Eq. (2.2) represents the effects of couple stresses in an incompressible fluid. Here μ_1 is a material constant responsible for the couple stress property and has the dimension of momentum (MLT^{-1}). According to Stokes' theory, the rheological flow properties for an incompressible viscous couple stress fluid are characterized by the two constants μ and μ_1. Since the dimension of μ is $ML^{-1}T^{-1}$ and that of μ_1 is MLT^{-1}, the ratio $(\mu_1/\mu)^{1/2}$ has the dimension of length (see Appendix A for the constitutive equations for couple stress fluid).

The effect of couple stress are quite large for large values of the non-dimensional number $a = l/d$, where d is the typical dimension of the flow geometry and l is the material constant $l = (\mu_1/\mu)^{1/2}$. If l is a function of the molecular dimensions of the liquid, it will vary greatly for different liquids. For example, the length of a polymer chain may be a million times the diameter of water molecule. Therefore, one may expect that couple stresses appear in noticeable magnitudes in liquids with large molecules.

The phenomenological equations relating the fluxes of heat $\mathbf{J}_T$ and matter $\mathbf{J}_C$ to the thermal and solute gradients present in binary mixture may be written as (De Groot and Mazur [22])

$$\mathbf{J}_T = -k\nabla T - \rho T S(\partial\lambda/\partial S)D'\nabla S \tag{2.3}$$

$$\mathbf{J}_C = -\rho D\left[S_T S(1-S)\nabla T + \nabla S\right] \tag{2.4}$$

where k is the thermal conductivity, D diffusion constant, ρ density, λ the chemical potential of the solute. T and S are temperature and concentration respectively. The terms containing the Soret coefficient S_T and Dufour coefficient D' give rise to interaction between the thermal and solute fields even when the mixture is at rest. Since $D' = S_T D$ (the Onsager reciprocal principle), the Dufour coefficient is of order of magnitude smaller than the Soret coefficient in liquids, and the corresponding contribution to the heat flux may be neglected (see e.g. Hurle and Jakeman [9] and Straughan and Hutter [23]). In the present paper we are dealing with liquids, and Eq. (2.3) will be replaced by

$$\mathbf{J}_T = -k\nabla T \tag{2.5}$$

Hurle and Jakeman [9] treat the term $S(1-S)$ in the Soret effect as having a mean constant value, hence we here treat the Soret term to have a constant coefficient. From a mathematical viewpoint we may then without loss of generality, incorporating Eqs. (2.4) and (2.5), write the temperature and concentration equations, in the form

$$\frac{\partial T}{\partial t} + (\mathbf{q}\cdot\nabla)T = \kappa\nabla^2 T \tag{2.6}$$

$$\frac{\partial S}{\partial t} + (\mathbf{q}\cdot\nabla)S = D\nabla^2 S + D_1\nabla^2 T \tag{2.7}$$

Here D_1 quantifies the contribution to the mass flux due to temperature gradient. The equation of state is

$$\rho = \rho_0\left[1 - \beta_T(T - T_0) + \beta_S(S - S_0)\right] \tag{2.8}$$

The validity of the Oberbeck–Boussinesq approximation for the Navier–Stokes case has been proved recently by Rajagopal et al. [24]. There is need to seek the status of this approximation in the present case. The effect of suspended particles is to enhance the viscosity of the carrier fluid and does in any foreseeable way work to violate the Oberbeck–Boussinesq approximation. With this argument we accept the approximation for couple stress fluid.

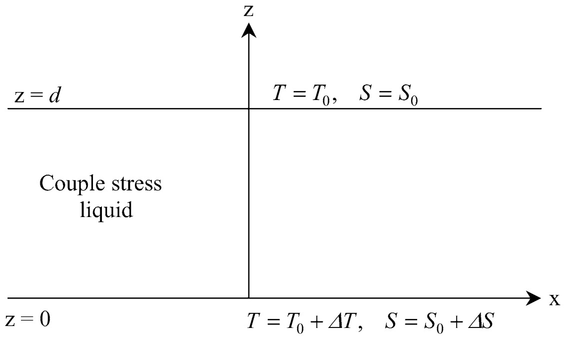

Fig. 1. Physical configuration.

We consider a horizontal couple stress liquid layer of thickness d confined between two parallel infinite stress free boundaries. The x-axis is taken along the lower boundary, and the z-axis vertically upward. The couple stress liquid is heated and salted from below. The temperature and concentration difference between the bounding walls being, ΔT and ΔS respectively. The schematic of the physical configuration is shown in Fig. 1.

2.2. Basic state

The basic state of the fluid is quiescent and is given by,

$$\mathbf{q}_b = (0,0,0), \qquad p = p_b(z), \qquad T = T_b(z)$$
$$S = S_b(z), \qquad \rho = \rho_b(z) \tag{2.9}$$

Using (2.9), Eqs. (2.1)–(2.2) and (2.6)–(2.8) yield

$$\frac{\mathrm{d}p_b}{\mathrm{d}z} = \rho_b g, \qquad \frac{\mathrm{d}^2 T_b}{\mathrm{d}z^2} = 0, \qquad \frac{\mathrm{d}^2 S_b}{\mathrm{d}z^2} = 0$$
$$\rho_b = \rho_0\big[1 - \beta_T(T_b - T_0) + \beta_S(S_b - S_0)\big] \tag{2.10}$$

2.3. Perturbed state

On the basic state we superpose perturbations in the form

$$\mathbf{q} = \mathbf{q}_b + \mathbf{q}', \qquad T = T_b(z) + T', \qquad S = S_b(z) + S'$$
$$p = p_b(z) + p', \qquad \rho = \rho_b(z) + \rho' \tag{2.11}$$

where primes indicate perturbations. Introducing (2.11) in Eqs. (2.1)–(2.2) and (2.6)–(2.8) and using basic state equations (2.10), we obtain

$$\nabla \cdot \mathbf{q}' = 0 \tag{2.12}$$
$$\rho_0\left[\frac{\partial \mathbf{q}'}{\partial t} + (\mathbf{q}' \cdot \nabla)\mathbf{q}'\right] = -\nabla p' + \mu\nabla^2\mathbf{q}' - \mu_1\nabla^4\mathbf{q}' + \rho'\mathbf{g} \tag{2.13}$$
$$\frac{\partial T'}{\partial t} + (\mathbf{q}' \cdot \nabla)T' + w'\frac{\partial T_b}{\partial z} = \kappa\nabla^2 T' \tag{2.14}$$
$$\frac{\partial S'}{\partial t} + (\mathbf{q}' \cdot \nabla)S' + w'\frac{\partial S_b}{\partial z} = D\nabla^2 S' + D_1\nabla^2 T' \tag{2.15}$$
$$\rho' = -\rho_0[\beta_T T' - \beta_S S'] \tag{2.16}$$

We consider only two-dimensional disturbances and define stream function ψ by

$$(u', w') = \left(\frac{\partial\psi}{\partial z}, -\frac{\partial\psi}{\partial x}\right) \tag{2.17}$$

which satisfy the continuity Eq. (2.12).

Eliminating pressure from (2.13), introducing the stream function ψ and non-dimensionalising the resulting equation as well as Eqs. (2.14) and (2.15) using the following non-dimensional parameters

$$(x^*, z^*) = \left(\frac{x}{d}, \frac{z}{d}\right), \qquad t^* = \frac{t}{d^2/\kappa}, \qquad \psi^* = \frac{\psi}{\kappa}$$
$$T^* = \frac{T'}{\Delta T}, \qquad S^* = \frac{S'}{\Delta S} \tag{2.18}$$

we obtain

$$\frac{1}{Pr}\frac{\partial}{\partial t}(\nabla^2\psi) = -R\frac{\partial T}{\partial x} + R_S\frac{\partial S}{\partial x} + \nabla^4\psi - C\nabla^6\psi + \frac{1}{Pr}\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,z)} \tag{2.19}$$
$$\frac{\partial T}{\partial t} = -\frac{\partial\psi}{\partial x} + \nabla^2 T + \frac{\partial(\psi, T)}{\partial(x,z)} \tag{2.20}$$
$$\frac{\partial S}{\partial t} = -\frac{\partial\psi}{\partial x} + \tau\nabla^2 S + D_S\frac{R}{R_S}\nabla^2 T + \frac{\partial(\psi, S)}{\partial(x,z)} \tag{2.21}$$

where

$Pr = \dfrac{\nu}{\kappa}$,	Prandtl number
$R = \dfrac{\beta_T g \Delta T d^3}{\nu\kappa}$,	thermal Rayleigh number
$R_S = \dfrac{\beta_S g \Delta S d^3}{\nu\kappa}$,	solute Rayleigh number
$C = \dfrac{\mu_1}{\mu d^2}$,	couple stress parameter
$\tau = \dfrac{D}{\kappa}$,	diffusivity ratio
$D_S = \dfrac{D_1}{\kappa}\dfrac{\beta_S}{\beta_T}$,	Soret parameter

and the asterisks have been dropped for simplicity. Eqs. (2.19)–(2.21) are solved for stress-free, isothermal, vanishing couple-stress boundary conditions, namely,

$$\psi = \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^4\psi}{\partial z^4} = T = S = 0 \quad \text{at } z = 0, 1 \tag{2.22}$$

3. Linear stability theory

In this section, we discuss the linear stability analysis, which is very useful in the local non-linear stability analysis discussed in the next section. To make this study we neglect the Jacobians in Eqs. (2.19)–(2.21) and assume the solutions to be periodic waves of the form

$$\begin{bmatrix} \psi \\ T \\ S \end{bmatrix} = e^{\sigma t}\begin{bmatrix} \Psi\sin(\pi\alpha x) \\ \Theta\cos(\pi\alpha x) \\ \Phi\cos(\pi\alpha x) \end{bmatrix}\sin(\pi z) \tag{3.1}$$

where σ is the growth rate and in general a complex quantity ($\sigma = \sigma_r + i\sigma_i$), α is horizontal wavenumber.

Substituting Eqs. (3.1) in the linearized version of Eqs. (2.19)–(2.21), we get

$$\left(\frac{\sigma}{Pr}+\eta K^2\right)K^2\Psi=-R\pi\alpha\Theta+R_S\pi\alpha\Phi \tag{3.2}$$

$$\left(\sigma+K^2\right)\Theta=-\pi\alpha\Psi \tag{3.3}$$

$$\left(\sigma+\tau K^2\right)\Phi=-\pi\alpha\Psi-D_S\frac{R}{R_S}K^2\Theta \tag{3.4}$$

where $\eta=1+CK^2$, $K^2=\pi^2(\alpha^2+1)$ and η is representative of the viscosity of the fluid. In the case of Newtonian fluids we have $\eta=1$. Analyzing the expression for η it is obvious that the suspended particles add to the viscosity in conformity with the Einstein's observation. The enforcement of Oberbeck–Boussinesq approximation in suspensions is better than in the Newtonian fluid case. This is some kind of justification for the assumption of Oberbeck–Boussinesq approximation in suspensions. For non-trivial solution of Ψ, Θ and Φ, we require

$$\begin{aligned}R=&\left\{\left(\sigma+K^2\right)\left(\sigma+\tau K^2\right)\left(\frac{\sigma}{Pr}+\eta K^2\right)K^2\right.\\&\left.+R_S\pi^2\alpha^2\left(\sigma+K^2\right)\right\}\\&\times\left\{\pi^2\alpha^2\left[\sigma+K^2(D_S+\tau)\right]\right\}^{-1}\end{aligned} \tag{3.5}$$

3.1. Marginal state

If σ is real, then marginal stability occurs when $\sigma=0$. Then Eq. (3.5) gives the stationary Rayleigh number R^{st} at the margin of stability, in the form

$$R^{\mathrm{st}}=\frac{\eta\tau K^6}{\pi^2\alpha^2(D_S+\tau)}+\frac{R_S}{(D_S+\tau)} \tag{3.6}$$

The minimum value of the Rayleigh number R^{st} occurs at the critical wavenumber $\alpha=\alpha_c$ where α_c satisfies the equation

$$3C\pi^2\left(\alpha^2\right)^2+2\left(1+C\pi^2\right)\alpha^2-\left(1+C\pi^2\right)=0 \tag{3.7}$$

We note that the critical wavenumber α_c depends on the couple stress parameter C. In the absence of Soret effect, the expression for R^{st} reduces to

$$R^{\mathrm{st}}=\frac{\eta K^6}{\pi^2\alpha^2}+\frac{R_S}{\tau} \tag{3.8}$$

Further in the case of single component system, $R_S=0$, Eq. (3.8) gives

$$R^{\mathrm{st}}=\frac{\eta K^6}{\pi^2\alpha^2} \tag{3.9}$$

with the critical value given by,

$$R_c^{\mathrm{st}}=\frac{\pi^4(1+\alpha_c^2)^3[1+C\pi^2(1+\alpha_c^2)]}{\alpha_c^2} \tag{3.10}$$

and the critical wavenumber α_c is to be obtained from Eq. (3.7). These are exactly the values given by Siddheshwar and Pranesh [19] for a single component system. Further when $C=0$, i.e., in the absence of couple stresses, we get the classical results $\alpha_c^2=0.5$ and $R_c^{\mathrm{st}}=657.5$ for clear viscous fluid (see e.g., Chandrasekhar [25]).

3.2. Oscillatory state

It is well known that the oscillatory motions are possible only if some additional constraints like rotation, salinity gradient and magnetic field are present. For the oscillatory mode $\sigma_r=0$ and $\sigma_i\neq 0$. We put $\sigma=i\omega$ (ω is real) in Eq. (3.5) and rearrange the terms to get the oscillatory Rayleigh number R^{osc} at the margin of stability, in the form

$$\begin{aligned}R^{\mathrm{osc}}=&\left\{(1+\tau)\left[\eta^2Pr+(1+\tau)\eta+\frac{\tau}{Pr}\right]K^6\right.\\&\left.+\pi^2\alpha^2R_S(\eta Pr+\tau)\right\}\\&\times\left\{\pi^2\alpha^2(1+\eta Pr-D_S)\right\}^{-1}\end{aligned} \tag{3.11}$$

and the non-dimensional frequency ω^2 in the form

$$\begin{aligned}\omega^2=-&\left\{\eta K^6(1+\tau)(\tau+D_S)+\frac{\tau K^6}{Pr}(\tau+D_S)\right.\\&\left.-\eta\tau K^6+\pi^2\alpha^2(\tau+D_S-1)R_S\right\}\\&\times\left\{K^2\left[\eta+\frac{(1+\tau)}{Pr}-\frac{(\tau+D_S)}{Pr}\right]\right\}^{-1}\end{aligned} \tag{3.12}$$

In the next section we perform a non-linear stability analysis and express heat and mass transfer by conduction and convection and we observe the effect of suspended particles through C on it.

4. Non-linear theory

The finite amplitude analysis is carried out in this section via Fourier series representation for the stream function ψ, temperature T and concentration S in the form

$$\psi=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}(t)\sin(m\pi\alpha x)\sin(n\pi z) \tag{4.1}$$

$$T=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}B_{mn}(t)\cos(m\pi\alpha x)\sin(n\pi z) \tag{4.2}$$

$$S=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}E_{mn}(t)\cos(m\pi\alpha x)\sin(n\pi z) \tag{4.3}$$

Substituting Eqs. (4.1)–(4.3) into the set of coupled non-linear partial differential equations (2.19)–(2.21), we obtain a system of coupled, non-linear ordinary differential equations. It is however logical to use the observed fact that laboratory systems and practical situations involving suspensions often exhibit flows dominated by a few spatial harmonics. This allows one to choose a minimal representation from the above Fourier series. Further, these series are starting values for solving a more general non-linear convection problem.

The first effect of non-linearity is to distort the temperature and concentration fields through the interaction of ψ, T and also ψ, S. The distortion of these fields will corresponds to a change in the horizontal mean, i.e. a component of the form $\sin(2\pi z)$ will be generated. Thus a minimal Fourier series which describes the finite amplitude free convection is given by,

$$\psi = A_1(t)\sin(\pi\alpha x)\sin(\pi z) \tag{4.4}$$

$$T = B_1(t)\cos(\pi\alpha x)\sin(\pi z) + B_2(t)\sin(2\pi z) \tag{4.5}$$

$$S = E_1(t)\cos(\pi\alpha x)\sin(\pi z) + E_2(t)\sin(2\pi z) \tag{4.6}$$

where the amplitudes $A_1(t)$, $B_i(t)$ and $E_i(t)$ are to be determined from the dynamics of the system.

Substituting Eqs. (4.4)–(4.6) into Eqs. (2.19)–(2.21) and equating the coefficients of like terms we obtain the following non-linear autonomous system of differential equations

$$\dot{A}_1 = -\frac{R\pi\alpha Pr}{K^2}B_1 + \frac{R_S\pi\alpha Pr}{K^2}E_1 - K^2\eta PrA_1 \tag{4.7}$$

$$\dot{B}_1 = -\pi\alpha A_1 - K^2B_1 - \pi^2\alpha A_1B_2 \tag{4.8}$$

$$\dot{B}_2 = -4\pi^2B_2 + \frac{\pi^2\alpha}{2}A_1B_1 \tag{4.9}$$

$$\dot{E}_1 = -\pi\alpha A_1 - K^2\tau E_1 - K^2D_S\frac{R}{R_S}B_1 - \pi^2\alpha A_1E_2 \tag{4.10}$$

$$\dot{E}_2 = -4\pi^2\tau E_2 - 4\pi^2D_S\frac{R}{R_S}B_2 + \frac{\pi^2\alpha}{2}A_1E_1 \tag{4.11}$$

where the over dot denotes the time derivative.

The non-linear system of autonomous differential equations is not suitable to analytical treatment for the general time-dependent variable and we have to solve it using a numerical method. However, one can make qualitative predictions as discussed below. The system of Eqs. (4.7)–(4.11) is uniformly bounded in time and possesses many properties of the full problem. Like the original equations (2.12)–(2.16), Eqs. (4.7)–(4.11) must be dissipative. Thus volume in the phase space must contract. In order to prove volume contraction, we must show that velocity field has a constant negative divergence. Indeed,

$$\begin{aligned}&\frac{\partial\dot{A}_1}{\partial A_1} + \frac{\partial\dot{B}_1}{\partial B_1} + \frac{\partial\dot{B}_2}{\partial B_2} + \frac{\partial\dot{E}_1}{\partial E_1} + \frac{\partial\dot{E}_2}{\partial E_2}\\ &\quad = -\left[K^2(\eta Pr + 1 + \tau) + 4\pi^2(1+\tau)\right]\end{aligned} \tag{4.12}$$

which is always negative and therefore the system is bounded and dissipative. As a result, the trajectories are attracted to a set of measure zero in the phase space; in particular they may be attracted to a fixed point, a limit cycle or, perhaps, a strange attractor. From Eq. (4.12) we conclude that if a set of initial points in phase space occupies a region $V(0)$ at time $t = 0$, then after some time t, the end points of the corresponding trajectories will fill a volume

$$V(t) = V(0)\exp\left[-\left\{K^2(\eta Pr + 1 + \tau) + 4\pi^2(1+\tau)\right\}t\right]$$

This expression indicates that the volume decreases exponentially with time. We can also infer that, couple stresses and the Prandtl number tend to enhance dissipation.

Finally we note that the system of Eqs. (4.7)–(4.11) are invariant under the symmetry transformation $(A_1, B_1, B_2, E_1, E_2) \to (-A_1, -B_1, B_2, -E_1, -E_2)$.

From qualitative predictions we look into the possibility of an analytical solution. In the case of steady motions, Eqs. (4.7)–(4.11) can be solved in closed form. Setting the left-hand sides of Eqs. (4.7)–(4.11) equal to zero, we get

$$\eta K^4A_1 + R\pi\alpha B_1 - R_S\pi\alpha E_1 = 0 \tag{4.13}$$

$$\pi\alpha A_1 + K^2B_1 + \pi^2\alpha A_1B_2 = 0 \tag{4.14}$$

$$8B_2 - \alpha A_1B_1 = 0 \tag{4.15}$$

$$\pi\alpha A_1 + K^2D_S\frac{R}{R_S}B_1 + \tau K^2E_1 + \pi^2\alpha A_1E_2 = 0 \tag{4.16}$$

$$8\tau E_2 + 8D_S\frac{R}{R_S}B_2 - \alpha A_1E_1 = 0 \tag{4.17}$$

Writing B_1, B_2, E_1 and E_2 in terms of A_1, using Eqs. (4.13)–(4.17) and substituting these in (4.13), with $\frac{A_1^2}{8} = x$ we get

$$a_1x^2 + b_1x + c_1 = 0 \tag{4.18}$$

where

$$a_1 = \eta K^2\alpha^4 \tag{4.19}$$

$$b_1 = \frac{\eta K^4\alpha^2}{\pi^2}\left(1+\tau^2\right) + \frac{\alpha^4}{K^2}\left[\tau R_S + (D_S - 1)R\right] \tag{4.20}$$

$$c_1 = \frac{\eta\tau^2K^6}{\pi^4} + \frac{\tau^2\alpha^2}{\pi^2}\left[\frac{R_S}{\tau} - \left(\frac{D_S}{\tau} + 1\right)R\right] \tag{4.21}$$

The required root of Eq. (4.18) is,

$$x = \frac{1}{2a_1}\left\{-b_1 + \left(b_1^2 - 4a_1c_1\right)^{1/2}\right\} \tag{4.22}$$

When we let the radical in the above equation to vanish, we obtain the expression for finite amplitude Rayleigh number R^f, which characterizes the onset of finite amplitude steady motions. The finite amplitude Rayleigh number can be obtained in the form

$$R^f = \frac{1}{2x_1}\left\{-x_2 + \left(x_2^2 - 4x_1x_3\right)^{1/2}\right\} \tag{4.23}$$

where

$$x_1 = \frac{\alpha^8(D_S-1)^2}{K^4}$$

$$\begin{aligned}x_2 &= \frac{2\alpha^8\tau R_S}{K^4}(D_S - 1) + \frac{2\eta K^2\alpha^6}{\pi^2}\left(1+\tau^2\right)(D_S-1)\\ &\quad + \frac{4\eta\tau^2K^2\alpha^2}{\pi^2}\left(\frac{D_S}{\tau} + 1\right)\end{aligned}$$

$$\begin{aligned}x_3 &= \frac{\eta^2K^8\alpha^4}{\pi^4}\left(\tau^4 + \tau^2 + 1\right) + \frac{\alpha^8\tau^2R_S^2}{K^4}\\ &\quad + \frac{2\eta K^2\alpha^2}{\pi^2}\tau\left(1+\tau^2\right)R_S - \frac{\eta\tau^2K^2\alpha^6}{\pi^2}\frac{R_S}{\tau}\end{aligned}$$

5. Heat and mass transports

In the study of convection in fluids, the quantification of heat and mass transport is important. This is because the onset of convection, as Rayleigh number is increased, is more readily detected by its effect on the heat and mass transport. In the basic state, heat and mass transport is by conduction alone.

If H and J are the rate of heat and mass transport per unit area respectively, then

$$H = -\kappa \left\langle \frac{\partial T_{\text{total}}}{\partial z} \right\rangle_{z=0} \tag{5.1}$$

$$J = -D \left\langle \frac{\partial S_{\text{total}}}{\partial z} \right\rangle_{z=0} - D_1 \left\langle \frac{\partial T_{\text{total}}}{\partial z} \right\rangle_{z=0} \tag{5.2}$$

where the angular bracket corresponds to a horizontal average and

$$T_{\text{total}} = T_0 - \Delta T \frac{z}{d} + T(x, z, t) \tag{5.3}$$

$$S_{\text{total}} = S_0 - \Delta S \frac{z}{d} + S(x, z, t) \tag{5.4}$$

Substituting Eqs. (4.5) and (4.6) in Eqs. (5.3) and (5.4) respectively and using the resultant equations in (5.1) and (5.2), we get

$$H = \frac{\kappa \Delta T}{d}(1 - 2\pi B_2) \tag{5.5}$$

$$J = \frac{D \Delta S}{d}\left[(1 - 2\pi E_2) + \frac{D_S R}{\tau R_S}(1 - 2\pi B_2)\right] \tag{5.6}$$

The thermal and solute Nusselt numbers are defined by

$$Nu = \frac{H}{\kappa \Delta T / d} = 1 - 2\pi B_2 \tag{5.7}$$

$$Nu^S = \frac{J}{D \Delta S / d} = (1 - 2\pi E_2) + \frac{D_S R}{\tau R_S}(1 - 2\pi B_2) \tag{5.8}$$

Writing B_2 and E_2 in terms of A_1, using Eqs. (4.13)–(4.17), and substituting in Eqs. (5.7) and (5.8) respectively, we obtain

$$Nu = 1 + \frac{2\pi^2 \alpha^2 x}{K^2 + \pi^2 \alpha^2 x} \tag{5.9}$$

$$\begin{aligned} Nu^S = 1 &+ \frac{2\pi^2 \alpha^2 x}{\tau(\tau K^2 + \pi^2 \alpha^2 x \tau^{-1})} \\ &+ \frac{D_S x}{\tau R_S}\left[1 - \frac{2\pi^2 \alpha^2 (K^2 - \pi^2 \alpha^2 x \tau^{-1})}{(K^2 + \pi^2 \alpha^2 x)(\tau K^2 + \pi^2 \alpha^2 x \tau^{-1})}\right] \end{aligned} \tag{5.10}$$

The second term on the right-hand side of Eqs. (5.9) and second and third terms on the right-hand side of (5.10) represent the convective contribution to heat and mass transport respectively.

To know the transient behavior of thermal and solute Nusselt numbers the autonomous system of Eqs. (4.7)–(4.11) have been solved numerically using the Runge–Kutta method with appropriate initial conditions for different values of R^f, D_S and C and the expressions for the thermal and solute Nusselt numbers are computed. The variations of the thermal and solute Nusselt number with time are depicted in Figs. 6 and 7.

6. Results and discussion

The onset of double diffusive convection in a couple stress liquid layer in the presence of Soret effect is investigated using linear and non-linear theory. The linear theory gives the condition for the onset of stationary and oscillatory convection. The non-linear convection provides information regarding quantity of heat and mass transfer. The behavior of the system as a function of R_S depends upon the diffusivity ratio τ, the Soret parameter D_S, the couple stress parameter C and the Prandtl number Pr.

The variation of the critical Rayleigh number for stationary and oscillatory modes with couple stress parameter C for different values of the Soret parameter D_S, diffusivity ratio τ, solute Rayleigh number R_S and Prandtl number Pr is shown in Fig. 2(a)–(d). We observe from these figures that the effect of couple stresses are quite large for large values of the non-dimensional parameter C. We also find that the bigger value of C, the larger the value of the critical Rayleigh number R indicating that the effect of large couple stresses is to delay the onset of convection. Fig. 2(a) shows the effect of Soret parameter on the onset of convection for a fixed value of $Pr = 5.0$, $R_S = 10^3$ and $\tau = 0.3$. We observe that for positive Soret number, the convection first sets in as stationary mode and oscillatory motions are not possible. The system remains unstable with respect to monotonous disturbances and hence stationary mode is most dangerous and steady convection exists. Further an increase in the value of Soret parameter advances the onset of stationary convection. On the other hand it is interesting to note that, for negative Soret number the convection first sets in as oscillatory mode for small values of C. However, for $C > 0.33$ the trend reverses, that is convection first sets in as stationary mode. The effect of increasing (negatively) the value of Soret parameter is not much significant in the case of oscillatory mode.

The effect of the diffusivity ratio τ on the onset of convection for a fixed values of $Pr = 5$, $R_S = 10^3$ and $D_S = -0.1$ is shown in Fig. 2(b). We find that the effect of increasing τ is to increase the oscillatory Rayleigh number R_c^{osc} thereby making the system more stable for the parameters chosen for this case. Fig. 2(c) displays the effect of the solute Rayleigh number R_S on the onset of convection for a fixed value of the parameters $Pr = 5$, $\tau = 0.3$ and $D_S = -0.1$. We observe that the effect of increasing R_S is to increase the critical Rayleigh number. The effect of the Prandtl number on the onset of convection is shown in Fig. 2(d). It is found that the effect of increasing the value of Pr decreases the oscillatory Rayleigh number. It is important to note that for a fixed $R_S = 10^3$, $\tau = 0.3$ and $D_S = -0.1$, the oscillatory motions are possible when $Pr > 0.0475$.

The effect of solute Rayleigh number R_S, diffusivity ratio τ and Prandtl number Pr is insignificant for large couple stress parameter. However the Soret parameter has significant influence on the stability of the system for large C.

Fig. 3 indicates the variation of critical Rayleigh number of stationary and oscillatory modes with the solute Rayleigh number R_S for different values of the Soret parameter D_S and for a fixed values of $C = 0.45$, $\tau = 0.3$ and $Pr = 5.0$. It is interesting to note that for positive Soret parameter D_S, the convection first sets in as stationary mode. However for negative Soret parameter the instability sets in first as oscillatory mode when solute Rayleigh number is large e.g., when $D_S = -0.1$, the convection first sets in as stationary mode for small and moderate values of the solute Rayleigh number ($R_S \leqslant 10^3$ approximately) and for large R_S convection first sets in as oscillatory mode. Further as D_S is increased negatively, the range of R_S for which, the convection first sets in as stationary mode is reduced.

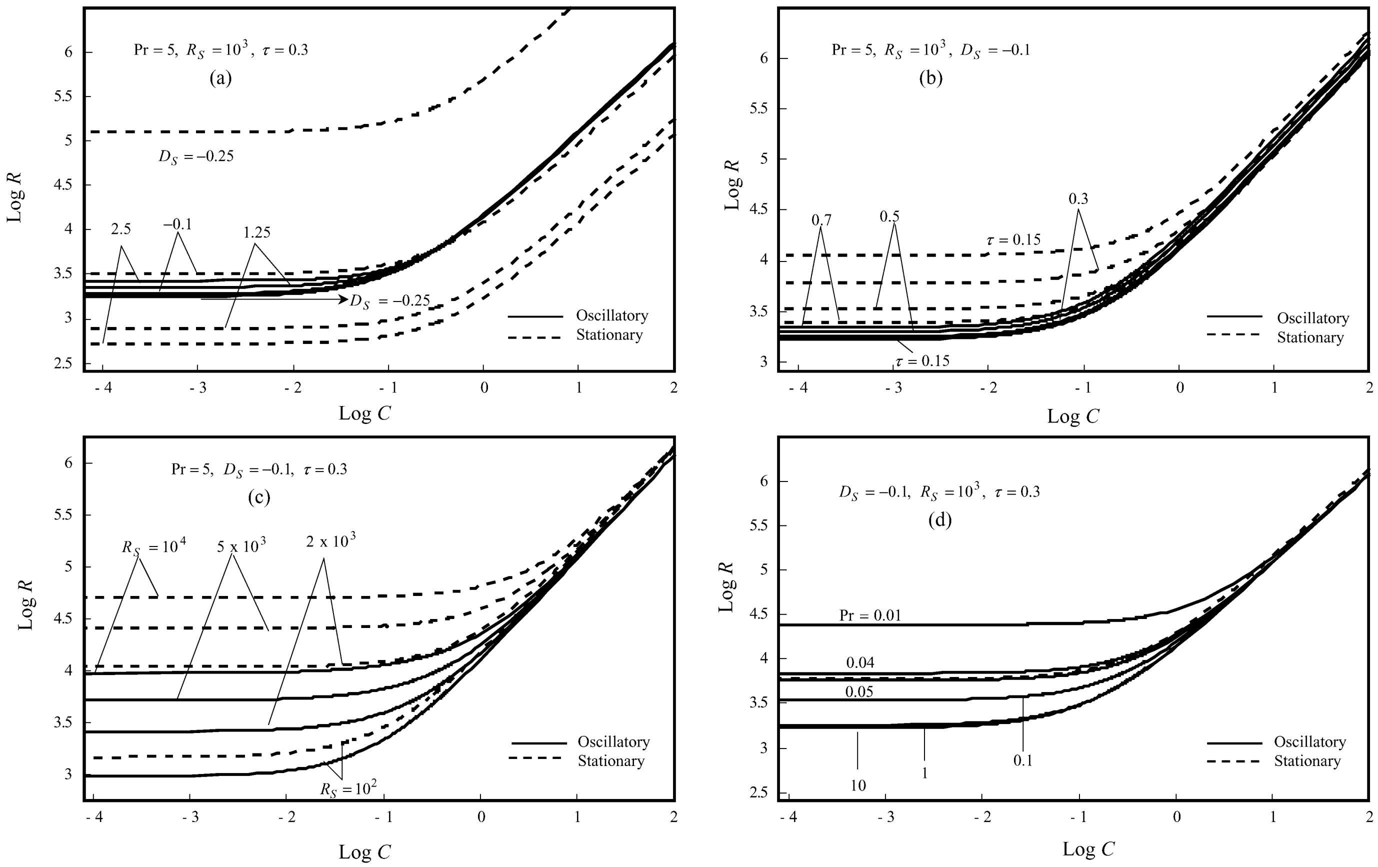

Fig. 2. Variation of critical Rayleigh number R with couple stress parameter C for different values of D_S, τ, R_S and Pr.

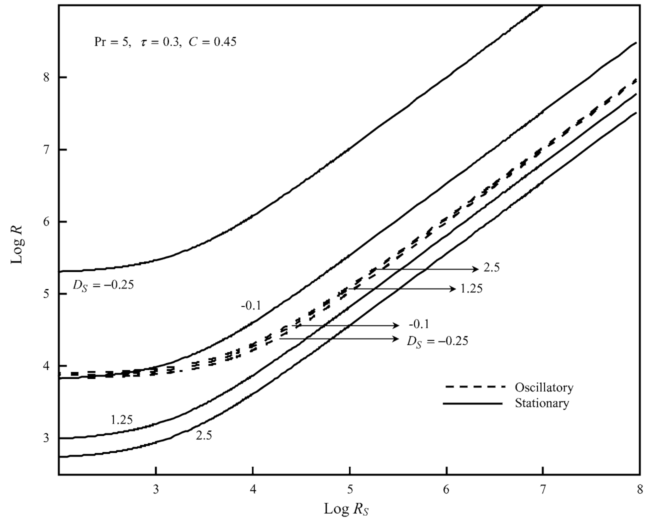

Fig. 3. Variation of critical Rayleigh number R with solute Rayleigh number R_S for different values of D_S.

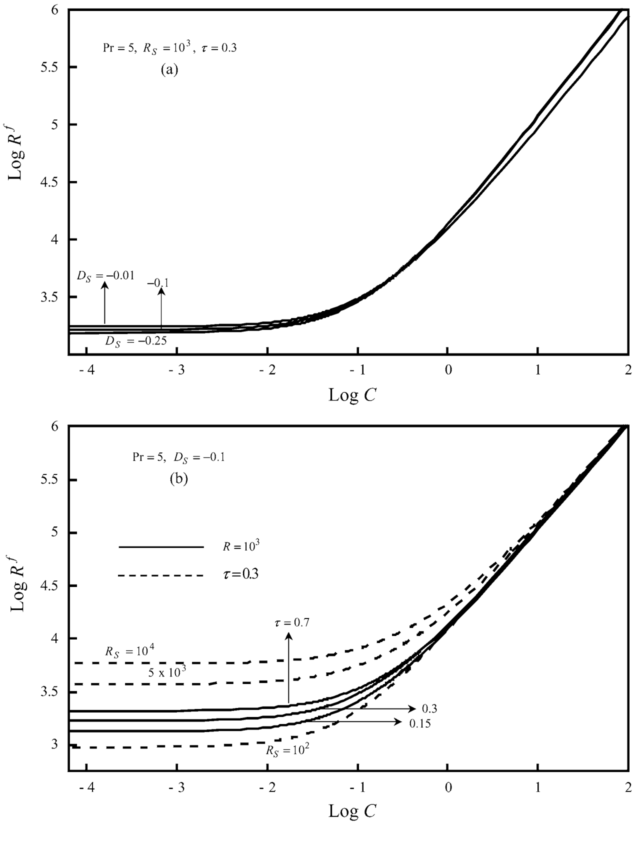

Fig. 4. Variation of finite amplitude critical Rayleigh number R^f with couple stress parameter C for different values of D_S, τ and R_S.

The variation of finite amplitude critical Rayleigh number R^f with couple stress parameter C for different values τ, D_S and R_S is shown in Fig. 4(a) and (b). It is important to note that the finite amplitude motions are ruled out for positive Soret parameter. The steady solutions are very interesting because they predict that a finite amplitude solution to the system is possible for negative Soret parameter. Further, the critical value of the Rayleigh number for which the finite amplitude motions occur

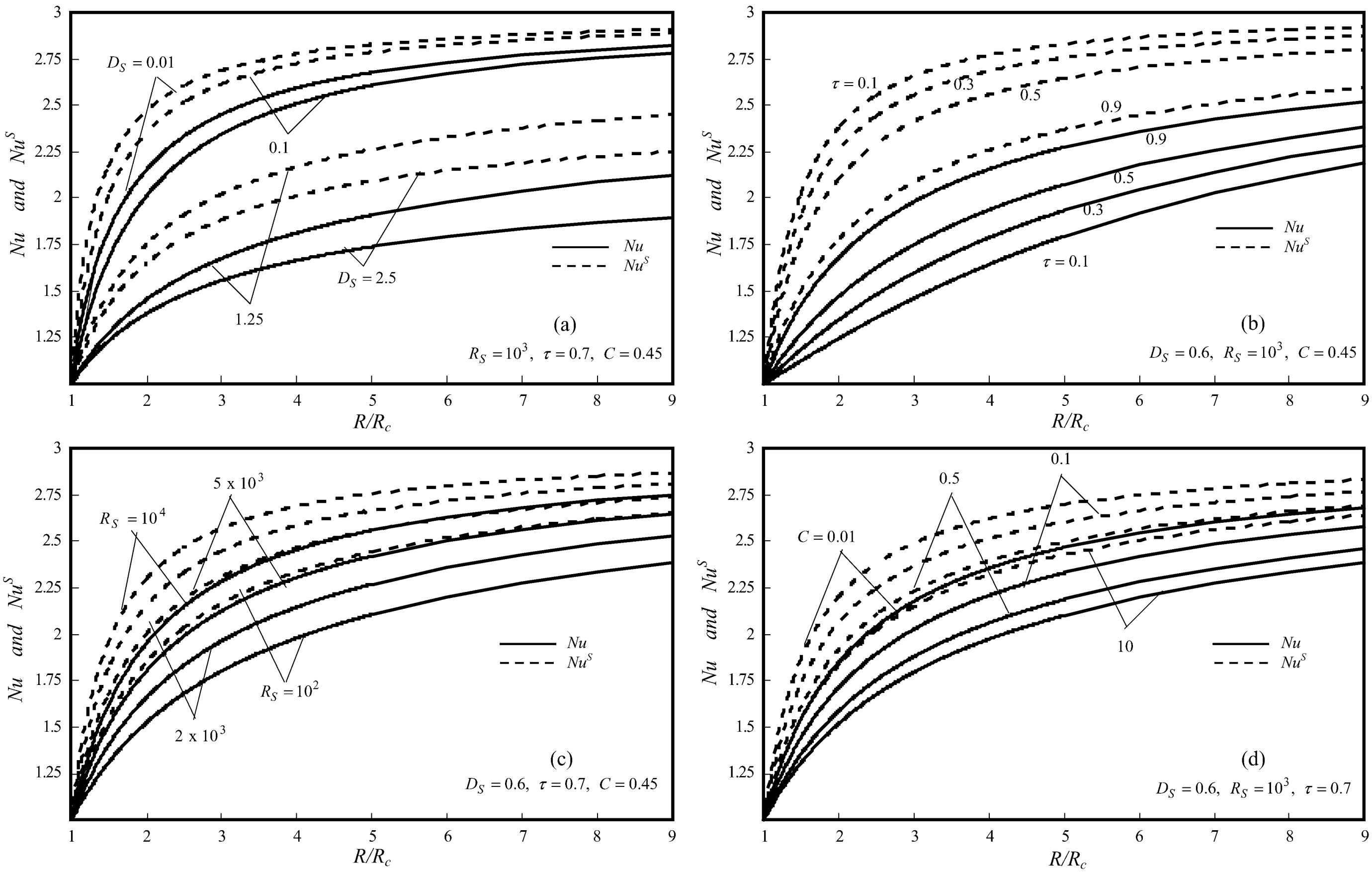

Fig. 5. Variation of Nusselt number with Rayleigh number R for different values of D_S, τ, R_S and C.

is less than the critical value of the Rayleigh number for stationary and oscillatory mode disturbances. Fig. 4(a) indicates that the effect of Soret parameter is very weak and is to enhance the instability for small C and this trend reverses for $C > 0.6$. The effect of diffusivity ratio τ and solute Rayleigh number R_S is found to be similar to their effects on oscillatory mode (Fig. 4(b)). That is the finite amplitude Rayleigh number R^f increases with increase in the value of solute Rayleigh number R_S and diffusivity ratio τ.

In the study of double diffusive convection the determination of heat and mass transport across the layer plays a very important role. Here, the onset of convection as the Rayleigh number is increased is more rapidly detected by its effect on the heat and mass transfer. The quantity of heat and mass transfer across the layer are given by Nu and Nu^S respectively, which represent the ratio of heat or mass transported across the layer to the heat or mass transported by conduction alone. Fig. 5(a)–(d) indicate the effect of R on the thermal and solute Nusselt numbers Nu and Nu^S for different values of the Soret parameter D_S, the diffusivity ratio τ, the solute Rayleigh number R_S and the couple stress parameter C. In each of these cases we observe that as Rayleigh number increases from one to four times of its critical value, the heat and mass transfer increase sharply and as Rayleigh number is increased further, they remain almost constant. It is also found that in each case the solute Nusselt number is above the thermal Nusselt number. We also note that the effect of increasing D_S and C is to decrease the values of Nu and Nu^S whereas that of R_S and negative Soret parameter is to increase Nu and Nu^S. Further the effect of increasing τ is to increase Nu whereas to decrease Nu^S. Although the presence of a stabilizing gradient of solute will inhibit the onset of convection, due to the strong finite amplitude motions, which exist for large Rayleigh numbers, tend to mix the solute and redistribute it so that the interior layers of the fluid are more neutrally stratified. As a consequence of that the inhibiting effect of solute gradient is greatly reduced and hence fluid will convect more and more heat and mass when R_S is increased.

To know the transient behavior, the variation of thermal and solute Nusselt numbers with time has been considered and is depicted in Figs. 6 and 7 for different values of R^f, D_S and C. It is clear that as R^f increases, the system becomes more unsteady and shows sensitivity to the initial conditions. However as time progresses we see that a steady state is reached via a transient state. Figs. 6(b) and 7(b) indicate that the effect of negative Soret parameter is to increase the amplitude of the oscillations and it takes longer time to reach the steady state when the negative Soret parameter is increased. The negative Soret parameter increases the heat and mass transport because it tends to increase the tendency of modes to exchange energy in chaotic way. Further the effect of couple stress is to decrease the amplitude of the oscillations of heat flux while it increase the amplitude of the oscillations of mass flux (Figs. 6(c) and 7(c)).

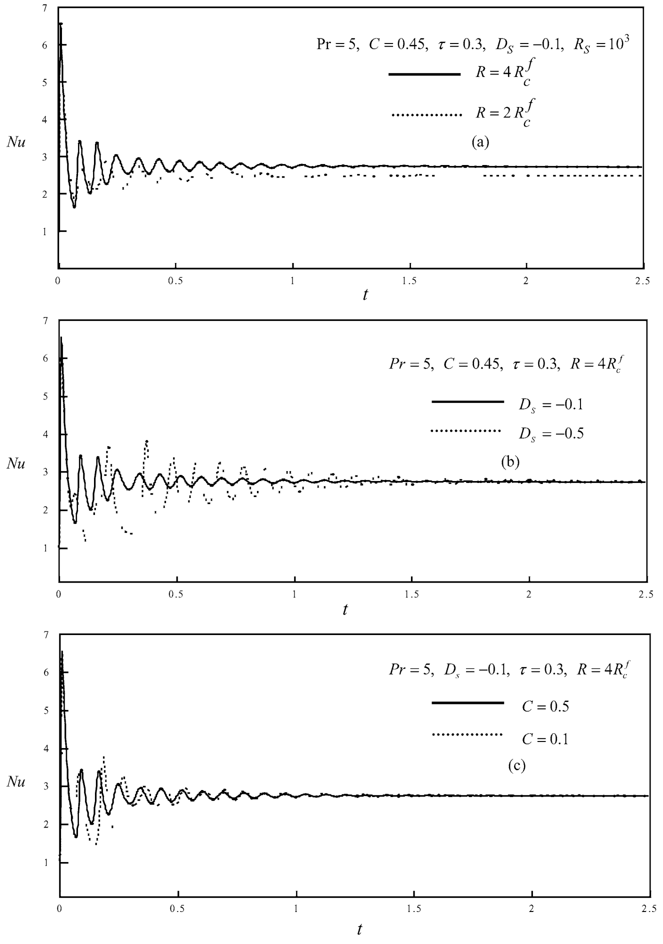

Fig. 6. Variation of thermal Nusselt number *Nu* with time *t* for different values of R, D_S and C.

7. Conclusions

The important results of the linear and non-linear double diffusive convection in couple stress liquids in the presence of Soret effect are summarized as follows:

(1) The effects of couple stresses are quite significant for large values of the non-dimensional parameter C and the large couple stress parameter delay the onset of convection.
(2) Convection first sets in as oscillatory mode for negative Soret parameter when the couple stress parameter C is small. However for large value of C, the convection first sets in as stationary mode.
(3) The finite amplitude motions are possible for only negative Soret parameter. The finite amplitude critical Rayleigh number is less than the critical Rayleigh number for stationary and oscillatory modes.
(4) The heat and mass transfer decreases with increase in the values of Soret parameter D_S and couple stress parameter C while both increase with increase in the value of the solute Rayleigh number R_S and negative Soret parameters. But with the increase in diffusivity ratio τ the heat transfer increases whereas the mass transfer decreases.

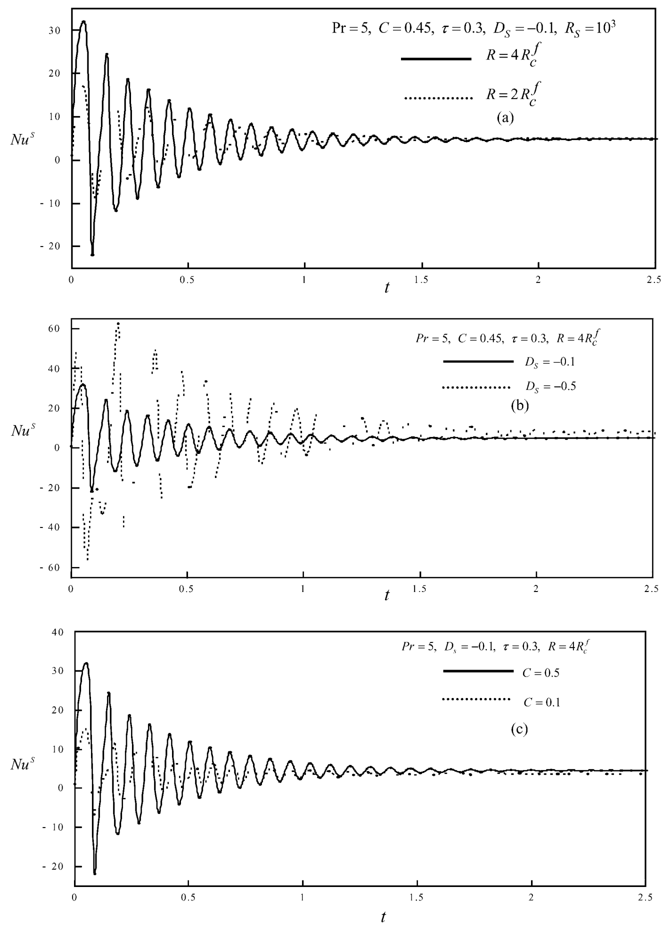

Fig. 7. Variation of solute Nusselt number Nu^S with time t for different values of R, D_S and C.

(5) The numerical results relating to *Nu* and Nu^S of unsteady case for large t compare well with the steady case values.

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Appendix A

The constitutive equations for couple stress fluids are listed in this appendix. The stress tensor T_{ij} can be decomposed into symmetric T_{ij}^s and antisymmetric T_{ij}^a parts, which are given by

$$T_{ij}^s = (-p + \lambda D_{kk})\delta_{ij} + 2\mu D_{ij} \tag{A.1}$$

$$T_{ij}^a = -2\eta W_{ij,kk} - \frac{\rho}{2}\varepsilon_{ijs} G_s \tag{A.2}$$

and the couple stress tensor

$$M_{ij} = 4\eta\omega_{j,i} + 4\eta'\omega_{i,j} \tag{A.3}$$

where the deformation tensor

$$D_{ij} = \frac{1}{2}(q_{i,j} + q_{j,i}) \tag{A.4}$$

the velocity tensor

$$W_{ij} = -\frac{1}{2}(q_{i,j} - q_{j,i}) \tag{A.5}$$

and

$$\omega_i = \frac{1}{2}\varepsilon_{ijk} q_{k,j} \tag{A.6}$$

Here q_i is the velocity field, λ and μ are material constants with the dimension of viscosity, and η and η' are material constants with the dimension of momentum.

The momentum equation governing the motion of an incompressible couple stress fluid, in the absence of body couples can be written in the vector form as

$$\rho_0\left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q}\right] = -\nabla p + \mu\nabla^2\mathbf{q} - \eta\nabla^4\mathbf{q} + \rho\mathbf{g} \tag{A.7}$$

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